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Author(s): Hidetoshi Konno*
Peter S. Lomdahl

*Institute of Information Sciences and Electronics, University of Tsukuba, Tsukuba, Ibaraki, 305-8573 Japan

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A Class of Nonlinear Birth-Death Stochastic Processes with Sub-Poissonian Statistics : Squeezed state of the particle number fluctuation

Hidetoshi Konno * and Peter S. Lomdahl

Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico, 87545, USA

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A class of Birth-death stochastic processes which gives sub-Poissonian statistics with two state variables is demonstrated. Mathematical structures for the appearance of sub-Poissonian statistics are also clarified in connection with contracted one variable models and their associated physical interactions.

KEYWORDS: classification of statistics, generalized birth-death process, multiple state variables, asymptotic evaluation, system size expansion, sub-Poissonian statistics, nonequilibrium open systems

§1. Introduction

The description of birth and death stochastic processes in open systems can be found in books by Feller[1], Risken[2], van Kampen[3], Gardiner[4], Montroll[5], Williams[6] and others. They describe many sophisticated stochastic processes and applications to physical, chemical, biological and engineering systems, most of them exhibiting super-Poissonian (SUPP) or Poissonian (P) statistics.

In a previous paper,[7] we have been investigating generalized birth-death stochastic processes of complicated nonlinear systems based on the system size expansion. Especially, we were interested in so called "rule-based dynamics" in nonequilibrium open systems wherein the chaotic movements of "particle"-like nonlinear excitations exist. One should note that in systems where creation/annihilation processes of "soliton" like excitations are taking place, the system is quite unstable and usually spatially inhomogeneous. Therefore, the asymptotic evaluation based on the system size expansion can not be applied directly to the original nonlinear partial differential equations such as a driven nonlinear Schrödinger equation,[8] the complex Ginzburg-Landau equation [9],[10] and so forth. However, since we are considering "rule-based dynamics" of "particle"-like excitations, the asymptotic evaluation [3],[11] by the system size can be applied. Actually, the relevant nonlinear kinetic equations associated with the "rule based dynamics" in those of a 1D driven non-

* *Permanent Address: Institute of Information Sciences and Electronics, University of Tsukuba, Tsukuba, Ibaraki, 305-8573 Japan*

linear Schrödinger equation,[8] 1D complex Ginzburg-Landau equation,[9),10] 1D Benney equation [12] and 1D Kuramoto-Sivashinsky equation [13] are stable and a unique stable steady-state exist.[14]

In conventional statistical mechanical studies, one makes notices of anomalous fluctuations associated with the nonequilibrium phase transitions near the onset of bifurcation points. Our approach is quite different from the conventional ones in that sense. We have shown the relevance of sub-Poissonian (SUBP) statistics for the creation/annihilation of "solitons" in systems described by "rule-based dynamics".

However, in the previous paper the relevant nonlinear system was restricted to contain a single state variable. Therefore, the feature of interactions among state variables are not easily seen from the given transitition probability $W(X,r)$. Also, the minimum value of the variance to the mean-to-variance ratio F is 1/3 among the simple examples shown in the previous paper. In our numerical experiments for a driven nonlinear Schödinger equation and the Benney equation, small F values less than 1/3 are obtained. It is not easy sometimes to understand what is the key factor among the possible origins determining the value of F such as higher order nonlinearity, memory effects due to feedback and so forth.

The aim of the present paper is complimentary to our previous paper and also to show generalized birth-death processes where SUBP statistics occur in open systems with two state variables and furthermore clarify physical mechanisms which give rise to the sub-Poissonian nature. The paper is organized as follows: Section 2 reviews typical generalized birth-death processes with one state variable. The key factor for determining the value of the variance to the mean value is clarified. Section 3 studies classification of statistics for a generalized stochastic process is performed with the use of the system size expansion method for two state variables based on the Haken-Zwanzing model. We will show several clear examples where SUBP statistics occurs. Here the physical mechanisms that affect SUBP statistics will be clarified. Section 4 discusses "squeezed states of particle number fluctuation" in comparison with the squeezed state of light and clarifies the concept of our "squeezed state of particle-number fluctuation" in nonlinear classical stochastic systems. Section 5 is devoted to concluding remarks.

§2. Effect of Nonlinearity Order upon Birth-Death Process with Single Variable

2.1 Examples with Lower Order Nonlinearity

Here we will start in reviewing the simple reaction which include the transition rates of lower order nonlinearity,

(A) A bi-molecular interaction

$$0 \to^{k_1} X \tag{2.1}$$

and

$$2X \rightarrow^{k_2} X \tag{2.2}$$

The transition probability (TP) $W(X, r, t)$ is expressed as

$$W(X, r, t) = k_1 X \delta_{r,1} + k_2 X(X-1)\delta_{r,-1}. \tag{2.3}$$

This is the simplest example for the TP in a nonlinear system. Surprisingly, this is the skeleton mechanism of creation/annihilation of "soliton"-like excitations in the Benney, the complex Ginzburg-Landau and the Kuramoto-Sivashinsky equation. Following the method of asymptotic evaluation with the use of system size expansion due to Kubo *et al* [11], let us scale the TP into

$$w(x, r, t) = \Omega^{-1} W(X, r, t) = b_0 \delta_{r,1} + a_2 x^2 \delta_{r,-1}\ , \tag{2.4}$$

where $x = \Omega^{-1} X$, Ω is the system size and b_0 and a_2 are the scaled coefficients. The method of system size expansion[3,11] gives the equation for the mean values and the variance around the mean:

$$y_t = c_1(y) \quad \text{and} \quad \sigma_t = 2c_1'(y)\sigma + c_2(y), \tag{2.5}$$

where $\langle \Omega^{-1} X \rangle = \langle x \rangle = y + \Omega^{-1} u_0 + \Omega^{-3/2} u_1$ and $c_k(x) = \sum_r r^k w(x, r, t)$.

The first and the second moments become

$$c_1(x) = b_0 - a_2 x^2 \quad \text{and} \quad c_2(x) = b_0 + a_2 x^2\ . \tag{2.6}$$

From the moments, we have the mean and the variance at the steady state as

$$y_s = \sqrt{\frac{b_0}{a_2}} \quad \text{and} \quad \sigma_s = \frac{1}{2} y_s\ . \tag{2.7}$$

This model thus gives SUBP statistics with the following variance-to-mean ratio F as

$$F = \frac{\sigma_s}{y_s} = \frac{1}{2} < 1\ , \tag{2.8}$$

which is independent on the transition rates (b_0 and a_2) of the TP.

(B) Tri-molecular reaction

Then consider the scaled TP with third order nonlinearity written as

$$w(x, r, t) = b_1 x \delta_{r,1} + a_3 x^3 \delta_{r,-1}\ . \tag{2.9}$$

This kind of nonlinearity appears frequently in practical applications since this TP gives the Ginzburg-Landau type equation with real number. One can easily obtain the mean value and the variance at the steady state as

$$y_s = \sqrt{\frac{b_1}{a_3}} \quad \text{and} \quad \sigma_s = \frac{y_s}{2}\ . \tag{2.10}$$

Therefore, we have SUBP statistics with

$$F = \frac{\sigma_s}{y_s} = \frac{1}{2} , \tag{2.11}$$

which is the same value as for the former case of TP in eq.(2.4) with second order nonlinearity. Also the value of F is independent of the transition rates b_1 and a_3. One should note that the TP in eq.(2.9) is a naive mathematical generalization from the TP in eq.(2.4) and that the physical significance of the value of F must be carefully interpreted.

Now let us assume the following physical process of triple molecular reactions:

$$A + X \rightleftharpoons^{k_1}_{k_2} 3X \tag{2.12}$$

and

$$B + X \rightarrow^{k_3} C \ . \tag{2.13}$$

The scaled TP should be

$$w(x, r, t) = b_1 x \delta_{r,1} + a_3 x^3 \delta_{r,-2} + a_1 x \delta_{r,-1} \ . \tag{2.14}$$

Since $c_1(x) = (b_1 - a_1)x - 2a_3x^3$ and $c_2(x) = (b_1 + a_1)x + 4a_3x^3$, the condition for obtaining the non-trivial steady state is $(b_1 - a_1) > 0$. With the use of the same scaling technique, the F value is obtained as

$$F = \frac{3(1 - \frac{1}{3}R)}{4(1 - R)} , \tag{2.15}$$

where

$$R = \frac{a_1}{b_1} \ (0 < R < 1) \ . \tag{2.16}$$

Note that the two limiting cases (a) $R \to 0$, $F \to 3/4$ and (b) $R \to 1$, $F \to \infty$. The divergence of F (also σ_s) in the case (b) is ascribed to a critical fluctuation near the transition point $R = 1 (a_1 = b_1)$. As seen in Fig.1, (i) SUBP statistics can appear in the range $0 < R < 1/3$; (ii) P statistics with $F = 1$ appears for $R = 1/3$ $(3a_1 = b_1)$; (iii) SUPP statistics can appear in range $1/3 < R < 1$. This example shows that there exists a physical or chemical process giving SUBP + P + SUPP statistics (*i.e.*, $3/4 < F < \infty$) with third order nonlinearity. An interesting aspect is that the value of F is independent of the value of a_3, but its lower bound is determined by the competing order of nonlinearities assciated with death and birth rates as shown below.

2.2 Effect of Higher Order Nonliearity

To exhibit the effect of higher order nonlinearity, let us take the scaled TP

$$w(x, r) = bx^n \delta_{r,1} + ax^m \delta_{r,-1} \ \ (n < m) \ . \tag{2.17}$$

This generalized TP includes (2.4) and (2.9). One can easily show from $y_s = (b/a)^{\frac{1}{m-n}}$ and $\sigma_s = \frac{1}{m-n}(b/a)^{\frac{1}{m-n}}$ that the variance-to-mean ratio as

$$F = \frac{1}{m - n} , \tag{2.18}$$

which is independent of the values of the birth-death rates (a and b) and is determined by the difference $(m-n)$ of the orders of different kinds of nonlinearity. When one takes $n=1$ and $m=5$ (the fifth order nonlinear term is relevant), $F=1/4$. As the difference of $(m-n)$ increases, the value of F decreases.

How the situation changes when three different kinds of nonlinearity is incorporated. We have shown [7),8)] that in the case of the scaled TP as $w(x,r,t) = b_1\delta_{r,1} + a_2x^2\delta_{r,-1} + a_3x^3\delta_{r,-1}$, the F value is within the range $1/2 < F < 1$ and varies as a function of $R = b_1a_3/4a_2$. How is the situation when fifth order nonlinearity is incorporated to in addition to third order. We examine the scaled TP as

$$w(x,r,t) = b_1x\delta_{r,1} + a_3x^3\delta_{r,-1} + a_5x^5\delta_{r,-1} \ . \tag{2.19}$$

Since

$$c_1(x) = b_1x - a_3x^3 - a_5x^5 \quad \text{and} \quad c_2(x) = b_1x + a_3x^3 + a_5x^5 \ , \tag{2.20}$$

there is only the non-trivial steady-state value

$$y_s^2 = \frac{-a_3 + \sqrt{a_3^2 + 4b_1a_5}}{2a_5} \ , \tag{2.21}$$

on account of the fact that $a_2 > 0$. Thus we have

$$F = \frac{\sigma_s}{y_s} = \frac{4 + \frac{1}{R} + \sqrt{\frac{1}{R^2} + \frac{4}{R}}}{16 + \frac{4}{R}} \ , \tag{2.22}$$

where

$$R = \frac{4b_1a_5}{a_3^2} \ . \tag{2.23}$$

Note that the two limiting cases (a) $R \to 0$, $F \to 1/2$ (cf. the TP in eq.(2.17) with $m=3$ and $n=1$) and (b) $R \to \infty$, $F \to 1/4$ (cf. the TP in eq.(2.17) with $m=5$ and $n=1$). Therefore, the existence of SUBP statistics with $1/4 < F < 1/2$ shown in Fig.2 is easily verified.

As seen in this section, the competing higher order nonlinear terms seem to be relevant to reduce the value of F. But it is not clear what physical processes and/or what complex interactions (feedback) among multiple state variables occur. So we will consider a couple of illuminating examples with two state variables which can be classified into the generalized Haken-Zwanzing model.

§3. Generalized Birth-Death Process within Haken-Zwanzing Model

Generally speaking, the exact analysis is not possible except for a few simple examples. We need to adopt some asymptotic evaluation and numerical estimation of the parameters in general. A systematic method to perform the asymptotic evaluation is the system size expansion which has been developed by van Kampen [3)] and Kubo *et al.* [11)] The result is summarized as described below.

The stochastic process of state variable $\vec{X}$ which has the probability distribution $P(\vec{X},t)$ is described by the Master equation ;

$$\frac{\partial}{\partial t}P(\vec{X},t) = -\int(\Delta\vec{X})W(\vec{X}\to\vec{X}+\Delta\vec{X},t)P(\vec{X},t) \ ,$$

$$+\int(\Delta\vec{X})W(\vec{X}-\Delta\vec{X}\to\vec{X},t)P(\vec{X}-\Delta\vec{X},t) \ , \tag{3.1}$$

where $W(\vec{X}\to\vec{X}+\Delta\vec{X},t)$ is the transition probability (TP) from $\vec{X}$ to $\vec{X}+\Delta\vec{X}$, which is rewritten as $W(\vec{X},\Delta\vec{X},t)$ for abbreviation.

Taking into account the feedback from many body effects due to the interaction of particles and/or state variables, the transition probability (TP) per unit time at t, $W(\vec{X},\Delta\vec{X},t)$ in the Master equation may be expressed by

$$W(\vec{X},\Delta\vec{X},t) = \Omega w(\vec{x},\Delta\vec{X},t) \ , \tag{3.2}$$

where Ω is the system size and

$$\vec{x} = \Omega^{-1}\vec{X} \ . \tag{3.3}$$

The method of system size expansion [3),11)] gives the equations for the mean and the variance around the mean (cf. eqs.(102)-(104) in ref.11):

$$\frac{\partial}{\partial t}\vec{y} = c_1(\vec{y}) \ \ , \tag{3.4}$$

$$\frac{\partial}{\partial t}\sigma = 2c_1'(\vec{y})\sigma + c_2(\vec{y}) \tag{3.5}$$

and

$$\frac{\partial}{\partial t}\vec{u} = c_1'(\vec{y})u + \frac{1}{2}c_1''(y)\sigma \ , \tag{3.6}$$

where

$$\langle\Omega^{-1}\vec{X}\rangle = \langle\vec{x}\rangle = \vec{y} + \Omega^{-1}\vec{u} + O(\Omega^{-2}) \ . \tag{3.7}$$

and the moments are given by the formula;

$$c_k(\vec{x}) = \int d(\Delta\vec{X})(\Delta\vec{X})^k w(\vec{x},\Delta\vec{X},t) \ . \tag{3.8}$$

At the steady state, *i.e.* $\frac{\partial}{\partial t}\vec{y} = \frac{\partial}{\partial t}\sigma = 0$, the mean $\vec{y}_s$ is determined by

$$c_1(\vec{y}_s) = 0 \ . \tag{3.9}$$

In contrast to the one state variable case, the expression of the regression matrix K_s and the diffusion matrix D_s at the steady state must be calculated according to the formula

$$K_s(\vec{y}_s) = \frac{\partial c_1(\vec{y}_s)}{\partial \vec{y}_s} \tag{3.10}$$

and

$$D_s(\vec{y}_s) = c_2(\vec{y}_s). \tag{3.11}$$

There are a number of important relations among the regression matrix K_s, the diffusion matrix D_s , the covariance σ_s and the matrix of irreversible circulation α_s :

$$K_s\sigma_s + \sigma_s K_s^T + D_s = 0 \ , \tag{3.12}$$

$$\alpha_s = \frac{1}{2}(\sigma_s K_s^T - K_s\sigma_s) \tag{3.13}$$

and

$$\sigma_s = -\frac{1}{2}K_s^{-1}(D_s + 2\alpha_s) \ . \tag{3.14}$$

So the determination of the variance is sometimes cumbersome but the manipulations are straightforward.

We will consider cases where the relevant stochastic equations are classified into the Haken-Zwanzing model. The stochastic Haken-Zwanzing model is written as

$$\frac{d}{dt}X = u(X) - v(X)Y + F_X(t) \tag{3.15}$$

and

$$\frac{d}{dt}Y = w(X) - \gamma(X)Y + F_Y(t) \ , \tag{3.16}$$

where $u(X)$, $v(X)$, $w(X)$ and $\gamma(X)$ are nonlinear functions of X. In the usual framework of the Langevin approach, the following Gaussian "white noise" property of the fluctuating forces are assumed:

$$\langle F_X(t)\rangle = \langle F_Y(t)\rangle = 0 \ , \tag{3.17}$$

$$\langle F_X(t)F_X(t')\rangle = D_{XX}\delta(t-t') \quad , \quad \langle F_Y(t)F_Y(t')\rangle = D_{YY}\delta(t-t') \tag{3.18}$$

and

$$\langle F_X(t)F_Y(t')\rangle = D_{XY}\delta(t-t') \ . \tag{3.19}$$

Since we take the Master equation approach the correlation of the two kind of fluctuating forces should be incorporated into the TP. Actually, the off-diagonal part of the diffusion matrix is evaluated from the second moment $c_2(\vec{y}_s)$.

(A) *Logistic-Verhulst Model with Feedback* ($u(X) = k_1X, v(X) = k_2X, w(X) = k_3X$ and $\gamma(X) = k_4$) :

$$X \to^{k_1} 2X \tag{3.20}$$

$$X + Y \to^{k_2} C \ . \tag{3.21}$$

$$X \to^{k_3} Y \tag{3.22}$$

and

$$Y \to^{k_4} C \quad . \tag{3.23}$$

This interaction scheme corresponds to the scaled TP,

$$w(\vec{x}, \Delta\vec{X}, t) = b_1 x \delta_{\Delta X,1}\delta_{\Delta Y,0} + a_2 xy \delta_{\Delta X,-1}\delta_{\Delta Y,0} + b_3 x \delta_{\Delta X,0}\delta_{\Delta Y,1} + a_4 y \delta_{\Delta X,0}\delta_{\Delta Y,-1} \quad . \tag{3.24}$$

This model appears in many different kinds of fields such as physics, chemistry, biology, sociology, ecology and engineering.[17),21)]

By noticing the first and the second moment

$$c_1(x,y) = \begin{pmatrix} b_1 x - a_2 xy \\ b_3 x - a_4 y \end{pmatrix} \tag{3.25}$$

and

$$c_2(x,y) = \begin{pmatrix} b_1 x + a_2 xy \, , & 0 \\ 0 \, , & b_3 x + a_4 y \end{pmatrix} \, , \tag{3.26}$$

we have the non-trivial steady-state mean,

$$\vec{y}_s = (x_s, y_s)^T = (\frac{b_1 a_4}{a_2 b_3}, \frac{b_1}{a_2})^T \tag{3.27}$$

and the regression matrix around it,

$$K_s = \begin{pmatrix} 0 \, , & -\frac{b_1}{b_3} a_4 \\ b_3 \, , & -a_4 \end{pmatrix} . \tag{3.28}$$

Since the characteristic equation is $Det(\lambda E - K_s) = \lambda^2 + a_4\lambda + b_1 a_4 = 0$, the fixed point $\vec{y}_s$ is always stable. One should also note that the off-diagonal element of the diffusion matrix $D_s \equiv c_2(x_s, y_s)$ is zero. Namely, the two fluctuating forces of the corresponding Langevin equation are statistically independent.

With the use of the formula shown in Appendix A, we have the xx-component of the variance as

$$\sigma_s^{(xx)} = (1 + \frac{b_1}{b_3} + \frac{b_1}{a_4}) x_s \; . \tag{3.29}$$

Hence the variance-to-mean ratio becomes

$$F = \frac{\sigma_s^{(xx)}}{x_s} = 1 + \frac{b_1}{b_3} + \frac{b_1}{a_4} > 1 \; . \tag{3.30}$$

Thus F takes a value greater than 1 for any value of b_1, a_2, b_3, a_4 (*i.e.*, SUPP statistics) at the steady state x_s and y_s. In order to have SUBPstatistics, one must account for higher order nonlinear terms and/or other memory (feedback) effect.

(B) *Brusselator* ($u(X) = A - (B+1)X, v(X) = -X^2, w(X) = BX$ and $\gamma(X) = X^2$):

$$A \to X \tag{3.31}$$

$$2X + Y \to 3X \quad . \tag{3.32}$$

$$B + X \to Y + D \tag{3.33}$$

and

$$X \to E \quad . \tag{3.34}$$

The scaled TP in conjunction with the Brusselator becomes

$$w(\vec{x}, \Delta\vec{X}, t) = a\delta_{\Delta X,1}\delta_{\Delta Y,0} + x^2y\delta_{\Delta X,1}\delta_{\Delta Y,-1} + bx\delta_{\Delta X,-1}\delta_{\Delta Y,1} + x\delta_{\Delta X,-1}\delta_{\Delta Y,0} \, . \tag{3.35}$$

The first and the second moment take the form:

$$c_1(x,y) = \begin{pmatrix} a + x^2y - bx - x \\ -x^2y + bx \end{pmatrix} \tag{3.36}$$

and

$$c_2(x,y) = \begin{pmatrix} a + x^2y + bx + x \, , & -x^2y - bx \\ -x^2y - bx \, , & x^2y + bx \end{pmatrix} \, . \tag{3.37}$$

The value of the off-diagonal matrix element of the diffusion matrix $D_s = c_2(x_s, y_s)$ becomes negative for any values of x and y since the number density x and y are positive definite quantities.

The steady-state mean $\vec{y}_s$ becomes

$$\vec{y}_s = (x_s, y_s)^T = (a, \frac{b}{a})^T \, . \tag{3.38}$$

Since the regression matrix is

$$K_s = \begin{pmatrix} b-1 \, , & a^2 \\ -b \, , & -a^2 \end{pmatrix} \, , \tag{3.39}$$

the characteristic equation becomes $Det(\lambda E - K_s) = \lambda^2 + \Gamma\lambda + \Delta = 0$, where $\Gamma = -Tr(K_s) = 1 + a^2 - b$ and $\Delta = a^2$. When $\Gamma > 0$, the fixed point $\vec{y}_s$ in eq.(3.38) is stable. Within the stable region the xx component of variance $\sigma_s^{(xx)}$ becomes

$$\sigma_s^{(xx)} = \frac{a^3 + ab + a}{a^2 + 1 - b} \, . \tag{3.40}$$

Hence, we have only SUPP statistics for the Brusselator, *i.e.*

$$F = \frac{\sigma_s^{(xx)}}{x_s} = \frac{a^2 + 1 + b}{a^2 + 1 - b} > 1 \, . \tag{3.41}$$

Figures 3 and 4 show the variations of $\sigma_s^{(xx)}$ and F as a function of a for various values of $b = 1/2, 3/4$ and 1 within the stable region. One can see that the large variation of the variance $\sigma_s^{(xx)}$ does exist for small values of a $(0 < a < 1)$ for $1/2 < b < 1$. This example shows that (i) the negative value of the diffusion matrix $D_s = c_2(x_s, y_s)$ and (ii) the existence of third order nonlinearity (a tri-molecular reaction X^2Y) does not always work to give SUBP statistics. One should also note that we are considering the statistics in the stable "rule-based-dynamics" which has no direct connection with chemical oscillations, pattern formations and coherent excitations described by partial differential equations (PDEs). But there might exist indirect connection between the "rule-based-dynamics" of particle-like excitations in high dimensional chaotic systems and the phenomena described by PDEs. Therefore, the "rule-based -dynamics" applied to the chaotic complex oscillations in space-dependent Brusselator is our main interests and it might be subjected to a nonlinear stochastic process with global stability.

(C) *Creation and annihilation of "soliton"* (a high dimensional chaos) ($u(X) = k_3X - k_2X^2, v(X) = -k_1 - k_4X, w(X) = k_3X$ and $\gamma(X) = k_1 + k_4X$):

$$Y \to^{k_1} X \tag{3.42}$$

$$2X \to^{k_2} X \ . \tag{3.43}$$

and

$$X + Y \rightleftharpoons^{k_3}_{k_4} X \tag{3.44}$$

Here X denotes "soliton" and Y denotes "radiation". The scaled TP associated with the above "soliton"-"radiation" interaction scheme reduces to

$$w(\vec{x}, \Delta\vec{X}, t) = b_1 y \delta_{\Delta X,1}\delta_{\Delta Y,-1} + a_2 x^2 \delta_{\Delta X,-1}\delta_{\Delta Y,0} + b_3 x \delta_{\Delta X,1}\delta_{\Delta Y,1} + a_4 xy \delta_{\Delta X,-1}\delta_{\Delta Y,-1} \ . \tag{3.45}$$

The first and the second moment are

$$c_1(x,y) = \begin{pmatrix} b_1y - a_2x^2 + b_3x - a_4xy \\ -b_1y + b_3x - a_4xy \end{pmatrix} \tag{3.46}$$

and

$$c_2(x,y) = \begin{pmatrix} b_1y + a_2x^2 + b_3x + a_4xy \ , & -b_1y + b_3x + a_4xy \\ * \ , & b_1y + b_3x + a_4xy \end{pmatrix} \ . \tag{3.47}$$

There are three different steady-state solutions for $c_1(\vec{y}_s) = c_1(x_s, y_s) = 0$ (*i.e.*, the equation by contracting y_s becomes $x_s(a_2a_4x_s^2 + a_2b_1x_s - 2b_1b_3) = 0$). The non-trivial steady-state mean $\vec{y}_s$ among them is

$$\vec{y}_s = (x_s, y_s)^T = (\frac{b_1}{2a_4}\{ -1 + \sqrt{1 + 8\frac{b_3a_4}{b_1a_2}} \}, \frac{b_3x_s}{b_1 + a_4x_s})^T \ . \tag{3.48}$$

Since

$$K_s = \begin{pmatrix} -\frac{3}{2}a_2x_s \ , & (b_1 - a_4x_s) \\ \frac{1}{2}a_2x_s \ , & -(b_1 + a_4x_s) \end{pmatrix}, \tag{3.49}$$

the two coefficients $\Gamma = Tr(-K_s) = b_1 + a_4x_s + (3/2)a_2x_s$ and $\Delta = Det(-K_s) = b_1a_2x_s + 2a_2a_4x_s^2$ of the characteristic equation, $Det(\lambda E - K_s) = \lambda^2 + \Gamma\lambda + \Delta = 0$ take positive values (*i.e.*, $\Gamma > 0$

and $\Delta > 0$) for any values of b_1, a_2, b_3 and a_4. So the fixed point $\vec{y}_s$ is always stable. The xx component of the variance for this steady-state is expressed by

$$\sigma_s^{(xx)} = \frac{\{3b_1 + 2b_3 + (a_2 + 3a_4)x_s\}x_s}{2b_1 + (3a_2 + 2a_4)x_s} \ . \tag{3.50}$$

Hence we have

$$F = \frac{(3b_1 + 2b_3) + (a_2 + 3a_4)x_s}{2b_1 + (2a_4 + 3a_2)x_s} \ . \tag{3.51}$$

The expression is a complicated function in terms of the four birth-death rates (b_1, a_2, b_3 and a_4), and it is quite difficult to understand which statistics can appear. So let us rewrite (3.51) in the following form:

$$F = \frac{\sqrt{1 + 8R_1R_2} - 1 + 3R_1(1 + \sqrt{1 + 8R_1R_2}) + 4R_1R_2}{3(\sqrt{1 + 8R_1R_2} - 1) + 2(1 + \sqrt{1 + 8R_1R_2})R_1} \ , \tag{3.52}$$

where

$$R_1 = \frac{a_4}{a_2} \ \text{ and } \ R_2 = \frac{b_3}{b_1} \ (0 < R_1 < \infty \ \text{ and } \ 0 < R_2 < \infty) \ . \tag{3.53}$$

By noticing the limiting values (i) $R_1 \to 0$ and $R_2 \to \infty$, $F \to 2/3$, (ii) $R_2 \to 0$, $F \to 3/2$, ~~(iii) $R_1 \to \infty$, $F \to 3/2$ and (iv) $R_2 \to \infty$, $F \to \infty$, it is found that $2/3 < F < \infty$. Namely, SUBP~~ + P + SUPP statistics can appear depending on values of R_1 and R_2.

(D) *Edelstein's model* (a biochemical reaction) [22] ($u(X) = AX - X^2 + y_T, v(X) = X + 1, w(X) = 2Y_T$ and $\gamma(X) = X + B + 2$):

$$A + X \rightleftharpoons 2X \ , \tag{3.54}$$

$$X + Y \to C \tag{3.55}$$

and

$$C \rightleftharpoons Y + B \ , \tag{3.56}$$

where A and B are the concentrations of externally-controllable molecules, and Y and C are enzymes and their complex. Their concentrations satisfy the conservation law

$$Y + C = Y_T \ , \tag{3.57}$$

where Y_T is a constant and is fixed to be a nonzero value. The scaled TP corresponding to the above reaction scheme becomes

$$w(\vec{x}, \Delta\vec{X}, t) = ax\delta_{\Delta X,1}\delta_{\Delta Y,0} + x^2\delta_{\Delta X,-1}\delta_{\Delta Y,0} + xy\delta_{\Delta X,-1}\delta_{\Delta Y,-1}$$
$$+ by\delta_{\Delta X,0}\delta_{\Delta Y,-1} + (y_T - y)\delta_{\Delta X,1}\delta_{\Delta Y,2} \ . \tag{3.58}$$

The first and the second moment are

$$c_1(x, y) = \begin{pmatrix} ax - x^2 - xy + (y_T - y) \\ -yx - by + 2(y_T - y) \end{pmatrix} , \tag{3.59}$$

$$c_2(x,y) = \begin{pmatrix} a + x^2 + xy + (y_T - y) \ , & xy + 2(y_T - y) \\ * \ , & xy + by + 4(y_T - y) \end{pmatrix} \ . \tag{3.60}$$

The steady-state mean $\vec{y}_s$ is given by

$$\vec{y}_s = (x_s, y_s)^T = (x_s, \frac{2y_T}{x_s + b + 2})^T \ , \tag{3.61}$$

where x_s is given by the solution of the polynomial equation,

$$x_s^3 + (2 + b - a)x_s^2 + (y_T - a(2+b))x_s - by_T = 0 \ . \tag{3.62}$$

As is seen in the book by Glansdorff and Prigogine,[22] there are multiple non-trivial steady-states ($\vec{y}_{s1}$, $\vec{y}_{s2}$ and $\vec{y}_{s3}$) depending on values of a, b and y_T. Without loss of generality, one can assume that $x_{s1} < x_{s2} < x_{s3}$. When $b = 0.2$ and $y_T = 30$, the multiple-steady states appear in the range $a_{c1}(= 8.37) < a < a_{c2}(= 8.555)$ as shown in Fig.5. The two fixed points $\vec{y}_{s1}$ and $\vec{y}_{s3}$ are stable, and the other one $\vec{y}_{s2}$ is unstable. The variance $\sigma_s^{(xx)}$ is expressed in terms of x_s, y_s, a, b and y_T as

$$\begin{aligned} \sigma_s^{(xx)} &= \frac{1}{2}\frac{a + x_s^2 + x_s y_s + (y_T - y_s)}{\Gamma} \\ &+ \frac{1}{2}\frac{(x_s + b + 2)^2(a + x_s^2 + y_s x_s + y_T - y_s) + (x_s + 1)^2(y_s x_s + by_s + 4y_T - 4y_s)}{\Gamma\Delta} \\ &- \frac{2(x_s + 1)(x_s + b + 2)(y_s x_s + 2y_T - 2y_s)}{\Gamma\Delta} \ , \end{aligned} \tag{3.63}$$

where $\Gamma = 3x_s + b + 2 - a + y_s$ and $\Delta = 2x_s^2 + (2b + 4 - a)x_s + (b+1)y_s - a(b+2)$. When $b = 0.2$ and $y_T = 30$, $\sigma_s^{(xx)}$ and $F(= \sigma_s^{(xx)}/x_s)$ for two stable fixed points $\vec{y}_{s1}$ and $\vec{y}_{s3}$ are calculated as a function of a as shown in Figs.6 and 7. One can see in Fig. 6 that $\sigma_{s1}^{(xx)} \to \infty$ as $a \to a_{c2}$ and $\sigma_{s3}^{(xx)} \to \infty$ as $a \to a_{c1}$. Also one can see that (a) SUBP+P+SUPP statistics appears in both $\vec{y}_{s1}$ and $\vec{y}_{s3}$ branches; (b) SUBP statistics appear in the range far away from a_{c1} for x_{s3}, and the range far away from a_{c2} for x_{s1}. Considering the physical significance of the appearance SUBP statistics in biological and biochemical systems might be an interesting subject.

(E) *Tri-molecular reaction*

Now let us reconsider triple molecular reactions with two state variables (x, y) which can be reduced to the GL type equation as discussed in section 2.1(B) when the adiabatic elimination of y is applied. There exist at least two different types of interaction schemes with two state variables (x, y) in conjunction with the triple molecular reaction in eqs.(2.12) and (2.13) other than the Brusselator:

(i) Case I ($u(X) = k_1 X, v(X) = k_2 X, w(X) = k_3 X^2$ and $\gamma(X) = k_4 + k_2 X$);

$$X \to^{k_1} 2X \ , \tag{3.64}$$

$$X + Y \to^{k_2} B \ , \tag{3.65}$$

$$2X \rightarrow^{k_3} Y \quad , \tag{3.66}$$

and

$$Y \rightarrow^{k_4} C \quad . \tag{3.67}$$

The scaled TP is written as

$$w(\vec{x}, \Delta\vec{X}, t) = b_1 x \delta_{\Delta X,1}\delta_{\Delta Y,0} + a_2 xy \delta_{\Delta X,-1}\delta_{\Delta Y,-1} + b_3 x^2 \delta_{\Delta X,0}\delta_{\Delta Y,1} + a_4 y \delta_{\Delta X,0}\delta_{\Delta Y,-1} \; . \tag{3.68}$$

Since the first and the second moment are

$$c_1(x,y) = \begin{pmatrix} b_1 x - a_2 xy \\ b_3 x^2 - a_2 xy - a_4 y \end{pmatrix} \tag{3.69}$$

and

$$c_2(x,y) = \begin{pmatrix} b_1 x + a_2 xy \, , & a_2 xy \\ * \, , & a_2 xy + a_4 y + b_3 x^2 \end{pmatrix} \, , \tag{3.70}$$

the non-trivial steady-state

$$\vec{y}_s = (x_s, y_s)^T = (\frac{b_1}{2b_3}\{1 + \sqrt{1 + 4\frac{b_3 a_4}{b_1 a_2}}\}, \frac{b_1}{a_2})^T \, , \tag{3.71}$$

is always stable since $\Gamma = -Tr(K_s) = a_2 x_s + a_4 > 0$ and $\Delta = -Det(K_s) = a_2 x_s(2b_3 x_s - b_1) = a_2 x_s \sqrt{b_1^2 + 4b_1 b_3 a_4/a_2} > 0$. The expression of F is reduced to

$$F = \frac{\sigma_s^{(xx)}}{x_s} = \frac{1}{\sqrt{1+4R_1R_2}} + \frac{2R_1R_2}{1+4R_1R_2+\sqrt{1+4R_1R_2}} + \frac{2R_2}{2R_1R_2+1+\sqrt{1+4R_1R_2}} \; . \tag{3.72}$$

where

$$R_1 = a_4/b_1 \text{ and } R_2 = b_3/a_2 \; . \tag{3.73}$$

Note that there is no constaint for the paramaters (b_1, a_2, b_3 and a_4), $0 < R_1 < \infty$ and $0 < R_2 < \infty$. For special limiting cases are (a) $R_1 \to 0$ and $R_2 \to 0$, $F \to 1$; (b) $R_1 \to \infty$ and $R_2 \to \infty$, $F \to 0$; (c) $R_1 \to 0$ and $R_2 \to \infty$, $F \to \infty$. So it is shown that SUBP + P + SUPP statistics ($1/2 < F < \infty$) can appear depending the values of R_1 and R_2.

(ii) Case II ($u(X) = k_1 X, v(X) = k_2 X^2, w(X) = k_3 X$ and $\gamma(X) = k_4$);

$$X \rightarrow^{k_1} 2X \quad , \tag{3.74}$$

$$2X + Y \rightarrow^{k_2} X \quad , \tag{3.75}$$

$$X \rightarrow^{k_3} Y \quad , \tag{3.76}$$

and

$$Y \rightarrow^{k_4} C \quad . \tag{3.77}$$

The corresponding scaled TP is written as

$$w(\vec{x}, \Delta\vec{X}, t) = b_1 x \delta_{\Delta X,1}\delta_{\Delta Y,0} + a_2 x^2 y \delta_{\Delta X,-1}\delta_{\Delta Y,-1} + b_3 x^2 \delta_{\Delta X,0}\delta_{\Delta Y,1} + a_4 y \delta_{\Delta X,0}\delta_{\Delta Y,-1} \; . \tag{3.78}$$

Since the first and the second moment are

$$c_1(x,y) = \begin{pmatrix} b_1x - a_2x^2y \\ b_3x - a_2x^2y - a_4y \end{pmatrix} \tag{3.79}$$

and

$$c_2(x,y) = \begin{pmatrix} b_1x + a_2x^2y \ , & a_2x^2y \\ * \ , & a_2x^2y + b_3x + a_4y \end{pmatrix} \ , \tag{3.80}$$

the non-trivial steady-state exists under the condition

$$b_3 > b_1 \ , \tag{3.81}$$

which is given by

$$\vec{y}_s = (x_s, y_s)^T = (\sqrt{\frac{b_1a_4}{a_2(b_3-b_1)}} \}, \frac{b_1}{a_2x_s})^T \ . \tag{3.82}$$

Since $\Gamma = -Tr(K_s) = b_1 + a_4 + b_1a_4/(b_3 - b_1)$ and $\Delta = -Det(K_s) = a_2b_3 - 2b_1a_4 + b_3a_4$, the fixed point (3.80) is stable for

$$a_2b_3 > b_1a_4 \ . \tag{3.83}$$

The expression of F becomes

$$F = \frac{R_2 - 1}{(1+R_3)(R_2-1)+R_3} + \frac{R_3^2R_2(R_1^2R_2 - R_1R_3 + R_3^2)}{R_1^2(R_2-1)(R_2-1+R_2R_3)(R_1R_2-2R_3+R_2R_3)} \ . \tag{3.84}$$

where

$$R_1 = a_2/b_1 \ , R_2 = b_3/b_1 \text{ and } R_3 = a_4/b_1 \ (0 < \frac{R_3}{R_1R_2} < 1 \text{ and } 1 < R_2 < \infty) \ . \tag{3.85}$$

The range of variations for the parameters R_1, R_2 and R_3 are determined by the existence of the non-trivial steady (3.81) state and the stability requirement (3.83). It is shown that $0 < F < \infty$. Interestingly, the lower bound of F becomes zero. To see the nature clearly, let us show a few numerical examples in Figs.8(a) ($R_1 = 1, R_2 = 1.05$; $0 < R_3 < 1.05$), (b) ($R_1, R_2 = 10.5$; $0 < R_3 <$ 10.5) and (c) ($R_1 = 1, R_2 = 105$; $0 < R_3 < 105$); the minimum value of F (F_{min}) decreases and the position of R_3 taking the F_{min} increases as R_2 increases in keeping R_1 constant.

Hence the disapperance of discretized F value independent of birth-death rates and the possibility of the appearance of SUBP + P + SUPP statistics are shown associated with the complicated nonlinear two component models (C)-(F) with feedback.

§4. Squeezed State of Particle Number Fluctuation

We have demonstrated that the sub-Poissonian statistics is ubiquitous in physical, chemical, biological systems with the use of two variables nonlinear models as seen in section 3. Generally speaking, the appearance of the sub-Poissonian statists is determined by the highest order nonlinearity, the nature of the coupling among the state variables and the sign of the cross term of the diffusion matrix.

As pointed out in our previous papers,[7),8),12),14)] the physical situation we are concerned with is the noise-suppression mechanism and the squeezed state of particle number fluctuations such as coherent excitations like "soliton", "shock", "kink" or "hole"-like nonliner excitation as seen in section 3. In the case of the "squeezed state of light", the photon-number fluctuation is squeezed due to the anti-bunching of photons, which is a typical quantum effects.

The squeezed state of "soliton" number fluctuation [8),12),14)]

$$\sigma_{nn} \equiv \langle n^2 \rangle - \langle n \rangle^2 < \langle n \rangle \tag{4.1}$$

is also characterized by the anti-correlation of the "soliton"-"radiation" interaction.

But one must notice that in the case of the Logistic-Verhulst model, the sub-Poissonian nature is not realized even if the anti-correlation of the relaxation matrix and the off-diagonal element of the diffusion matrix is zero. Also one should note that in the case of the Brusselator in section 3 (B), the negative correlation of the off-diagonal part of the diffusion matrix does not work to provides the sub-Poissonian statistics.

§5. Concluding Remarks

We have developed a classification method of statistics for a generalized birth-death process with multiple state variables specifically in the two component systems within the Haken-Zwanzing model and demonstrated that the stochastic processes with SUBP statistics are ubiquitous in nonequilibrium open systems as shown and summarized in Tables I and II. The effects of higher order nonlinearity upon the value of the variance-to-mean ratio was also clarified. It was shown that higher order nonlinearity plays an important role for the occurrence of SUBP statistics. The necessary conditions for SUBP was also clarified.

The physical significance of the difference between the one state variable case and those of the corresponding two-state variables cases come from the feasibility condition of the adiabatic elimination of the second variable y. If we contract the state variable y by the adiabatic elimination for the Logistic-Verhulst model, the fluctuation of particle number x is subjected to the Poissonian statistics.

In order to obtain a small value of F which have been observed in numerical experiments in a driven nonlinear Schrödinger equation and the Benney equation,[14)] one needs to construct a model with multiple state variables and higher order nonlinearity. So to explain the situations where small values of F appears, one must take into account the negative correlation of the diffusion matrix, higher order nonlinearity and/or the non-Markovianity associated with the coherence of the quasi-particles such as "soliton", "shock", "kink", "hole"-like nonlinear excitation. The application of these ideas to systems with three state variable will be published in a separate paper. Finally, we would like to point out that Mori's scaling theory [19)] is also available in more sophisticated applications and discussions for non-uniform nonlinear systems with spatial diffusion.

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Appendix A

Expression of Variance in 2 Component System

Let B the relaxation matrix and D_s the diffusion matrix at the steady state :

$$B = -K_s = \begin{pmatrix} B_{xx} , & B_{xy} \\ B_{yx} , & B_{yy} \end{pmatrix} \text{ and } D_s = \begin{pmatrix} D_{xx} , & D_{xy} \\ D_{yx} , & D_{yy} \end{pmatrix} . \qquad (A1)$$

The xx component of the variance $\sigma_s^{(xx)}$ is given by

$$\sigma_s^{(xx)} = \frac{1}{2}\left(\frac{K_{xx}^{(1)}}{\Gamma} + \frac{K_{xx}^{(2)}}{\Gamma\Delta}\right) , \qquad (A2)$$

where

$$\Gamma = Tr(B) \ , \ \Delta = Det(B) \ , \qquad (A3)$$

$$K_{xx}^{(1)} = D_{xx} \quad \text{and} \quad K_{xx}^{(2)} = (B_{yy})^2 D_{xx} - 2B_{xy}B_{yy}D_{xy} + (B_{xy})^2 D_{yy} \ , \qquad (A4)$$

On the other hand, the irreversible circulation of fluctuation α_{xy} [20] is given by

$$\alpha_{xy} = \frac{1}{2}\frac{L_{xy}^{(1)}}{\Gamma} \qquad (A5)$$

where

$$L_{xy}^{(1)} = [B_{xy}D_{yy} - B_{yx}D_{xx} + (B_{xx} - B_{yy})D_{xy}] \ . \qquad (A6)$$

Table I

~~One-Component Nonlinear Model Giving Sub-Poissonian Statistics~~

Scaled Transition Probability	F-value	Statistics
$w(x,r,t) = b_0\delta_{r,1} + a_2x^2\delta_{r,-1}$:	$F = 1/2$:	SUBP
$w(x,r,t) = b_1x\delta_{r,1} + a_3x^3\delta_{r,-1}$:	$F = 1/2$:	SUBP
$w(x,r,t) = b_1x\delta_{r,1} + a_2x^2\delta_{r,-1} + a_3x^3\delta_{r,-1}$:	$1/2 < F < 1$:	SUBP
$w(x,r,t) = b_1x\delta_{r,1} + a_1x\delta_{r,-1} + a_3x^3\delta_{r,-2}$:	$3/4 < F < \infty$:	SUBP + P + SUPP
$w(x,r,t) = bx^n\delta_{r,1} + ax^m\delta_{r,-1}$:	$F = 1/(m-n) \ \ (m > n)$:	SUBP
$w(x,r,t) = b_1x\delta_{r,1} + a_3x^3\delta_{r,-1} + a_5x^5\delta_{r,-1}$:	$1/4 < F < 1/2$:	SUBP

P = Poissonian, SUBP = Sub-Poissonian and SUPP = Super-Poissonian Statistics

Table II

Statistics for various combinations of birth and death rates in the transition probability (TP) $w(x,y,r_x,r_y) = [\ a_0(m,n) + a_x(m,n)x + a_y(m,n)y + a_{xy}(m,n)xy + a_{2x}(m,n)x^2 + a_{2y}(m,n)y^2 + a_{x^2y}(m,n)x^2y\ ]\ \delta_{r_x,m}\delta_{r_y,n}$ in the 2-components nonlinear systems are listed in this table :

(A) Logistic-Verhulst Model

(B) Brusselator

(C) Soliton-radiation interaction Model

(D) Edelstein Model

(E) Tri-molecular Reaction Model (I)

(F) Tri-molecular Reaction Model (II)

Model	a_0	a_x	a_y	a_{2x}	a_{xy}	a_{2y}	a_{x^2y}	F-value	Statistics
(A)	*	(1,0) + (0,1)	(0,-1)	*	(-1,0)	*	*	$1 < F < \infty$	SUPP
(B)	(1,0)	(-1,0) + (-1,1)	*	*	*	*	(1,-1)	$1 < F < \infty$	SUPP
(C)	*	(1,1)	(1, -1)	(-1,0)	(-1,-1)	*	*	$2/3 < F < \infty$	SUBP + P + SUPP
(D)	(1,2)	(1,0)	(0,-1)	(-1,0)	(-1,-1)	*	*	$1/3 < F < \infty$	SUBP+P+SUPP
(E)	*	(1,0)	(0,-1)	(0,1)	(1,-1)	*	*	$1/2 < F < \infty$	SUBP + P + SUPP
(F)	*	(1,0) + (0,1)	(0,-1)	*	*	*	(-1,-1)	$0 < F < \infty$	SUBP + P + SUPP

P = Poissonian , SUBP = Sub-Poissonian and SUPP = Super-Poissonian Statistics

Figure Captions

Fig.1: The variation of the variance-to-mean ratio F in eq.(2.15) as a function of $R = a_1/b_1$ for a one-variable model of tri-molecular reaction. The variable range of R, $0 < R < 1$ is from the requirement for guarantee the existence of the non-trivial ($x_s \neq 0$) steady state.

Fig.2: The variation of the variance-to-mean ratio F in eq.(2.19) as a function of $R = 4b_1a_5/a_3^2$ for a one-variable model of tri-molecular reaction. The variable range of R is $0 < R < \infty$ since the non-trivial steady state is stable for any b_1, a_3 and a_5.

Fig.3: The variation of the variance σ_s^{xx} in eq.(3.40) as a function of a for the stochastic Brusselator for $b = 1/2, 3/4$ and 1. σ_s^{xx} for $b = 3/4$ and 1 tend to zero as a is approaching to zero.

Fig.4: The variation of the variance-to-mean ratio F in eq.(3.41) as a function of a for the stochastic Brusselator for $b = 1/2$ (dashed line), 3/4(dotted line) and 1(solid line). Although σ_s^{xx} for $b = 3/4$ and 1 tend to zero as a is approaching to zero, F take values greater than 1.

Fig.5: The variation of the steady states x_{sj} $(j = 1, 2, 3)$ $(x_{s1} < x_{s2} < x_{s3})$ as a function of a for the Edelstein model for $b = 0.2$ and $y_T = 30$. These three-steady state x_{sj} $(j = 1, 2, 3)$ coexist for $8.37 < a < 8.555$. The steady-state x_{s2}, which is depicted by the dotted line, is always unstable.

Fig.6: The variations of the variance σ_{s1} and σ_{s3} in eq.(3.63) as a function of a for the Edelstein model for $b = 0.2$ and $y_T = 30$. $\sigma_{s1}^{xx} < \sigma_{s3}^{xx}$ for $a < 8.35$, and $\sigma_{s1}^{xx} > \sigma_{s3}^{xx}$ for $a > 8.35$.

Fig.7: The variation of the variance-to-mean ratio F as a function of a for the Edelstein model for $b = 0.2$ and $y_T = 30$. F (at the branch x_{s1}) > F (at the branch x_{s3}) in the parameter range $a > 8.39$.

Fig.8: The variation of the variance-to-mean ratio F in eq.(3.84) for (a) $R_1 = 1, R_2 = 1.05$,

$0 < R_3 < 1.05$; (b) $R_1, R_2 = 10.5$, $0 < R_3 < 10.5$; and (c) $R_1 = 1, R_2 = 105$, $0 < R_3 < 105$). By observing these, one can see that the lower bound of F tends to zero as R_2 increases.

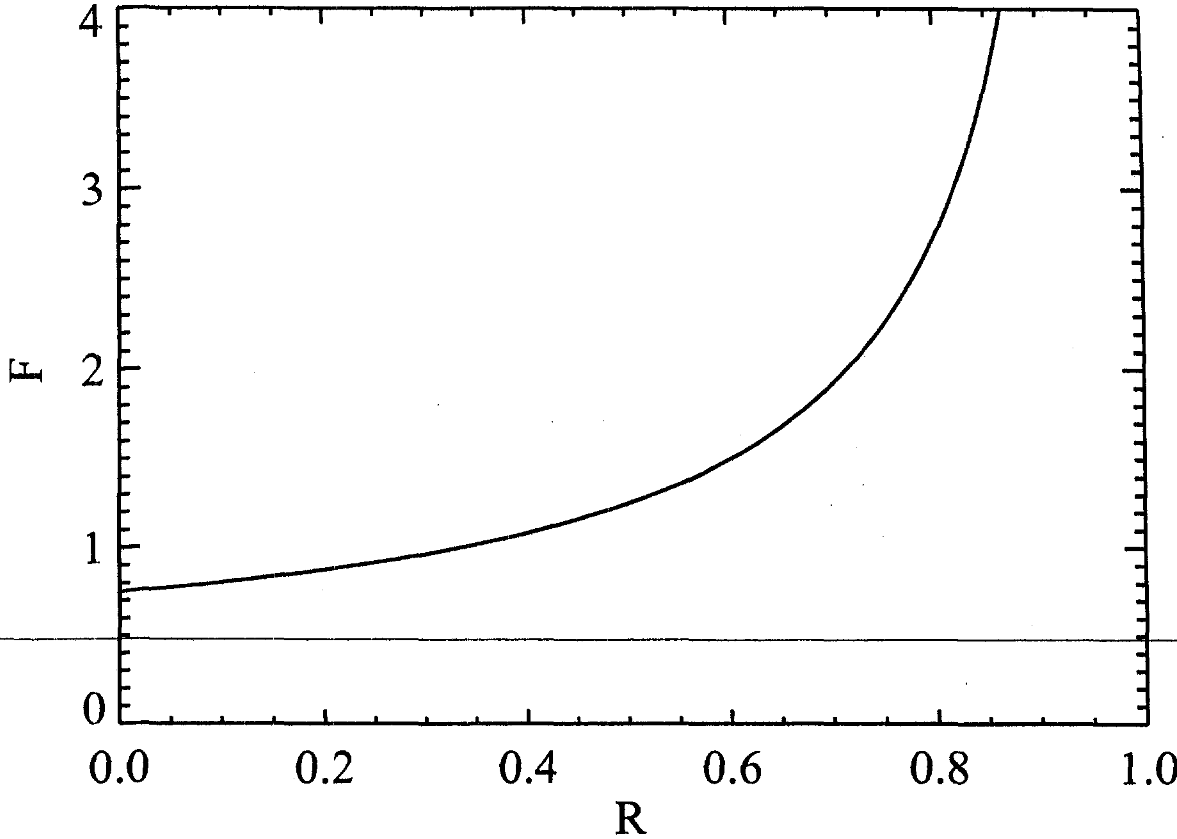

Fig. 1 Konno and Lomdahl $\times \frac{1}{2}$

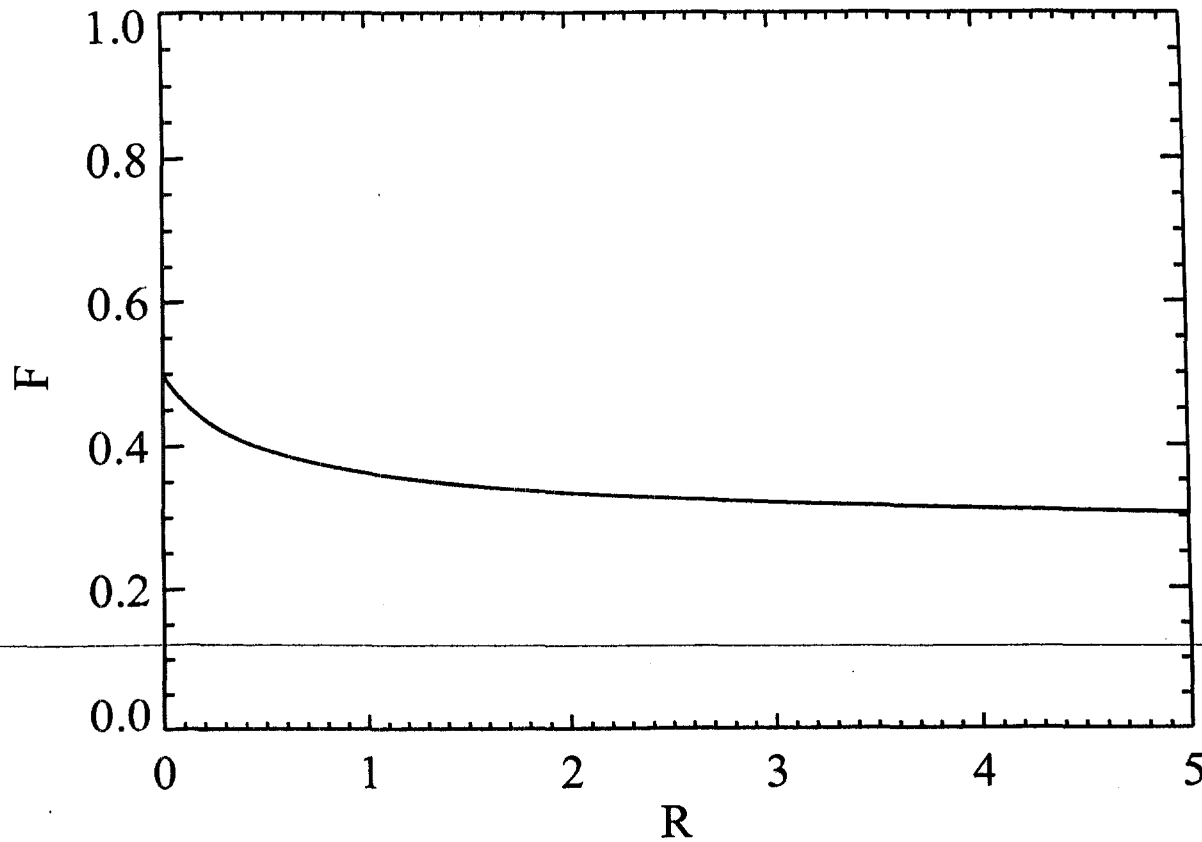

Fig. 2 Konno and Lomdahl $\times \frac{1}{2}$

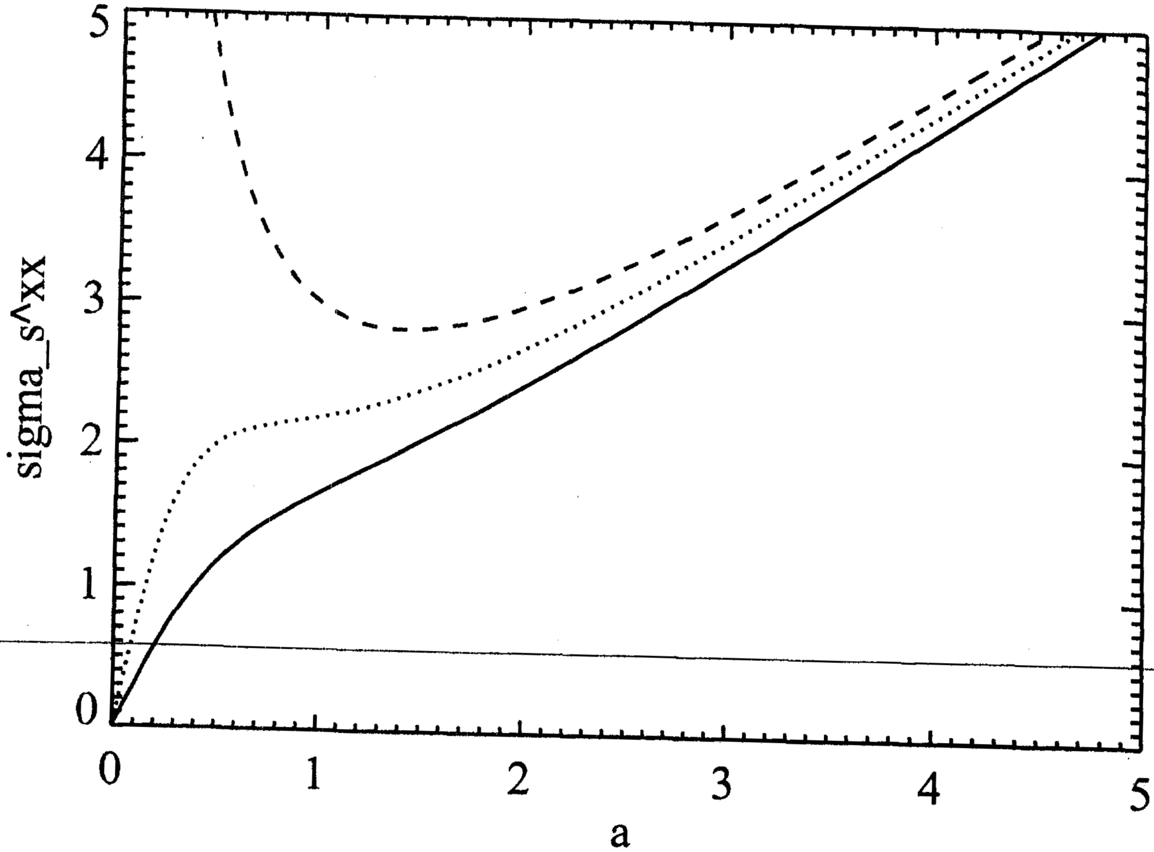

Fig. 3 Konno and Lomdahl $\times \frac{1}{2}$

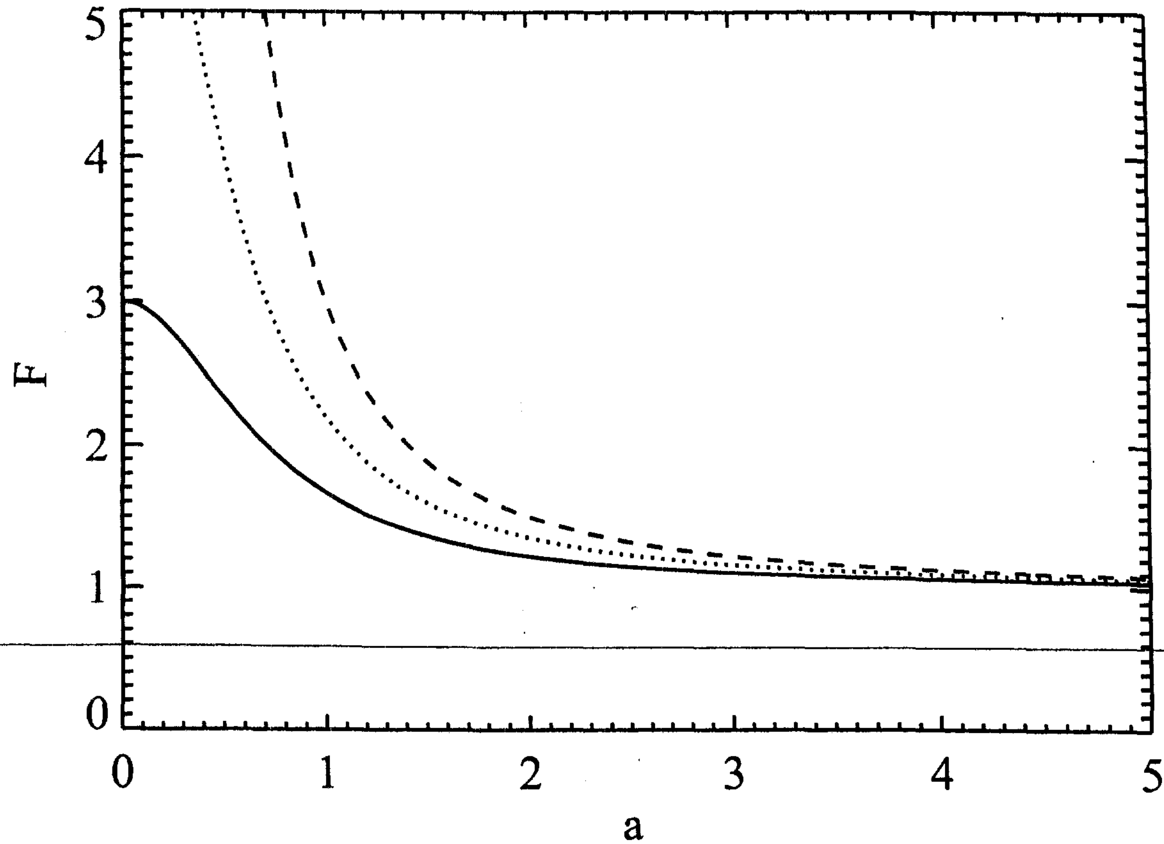

Fig. 4 Konno and Londahl × $\frac{1}{2}$

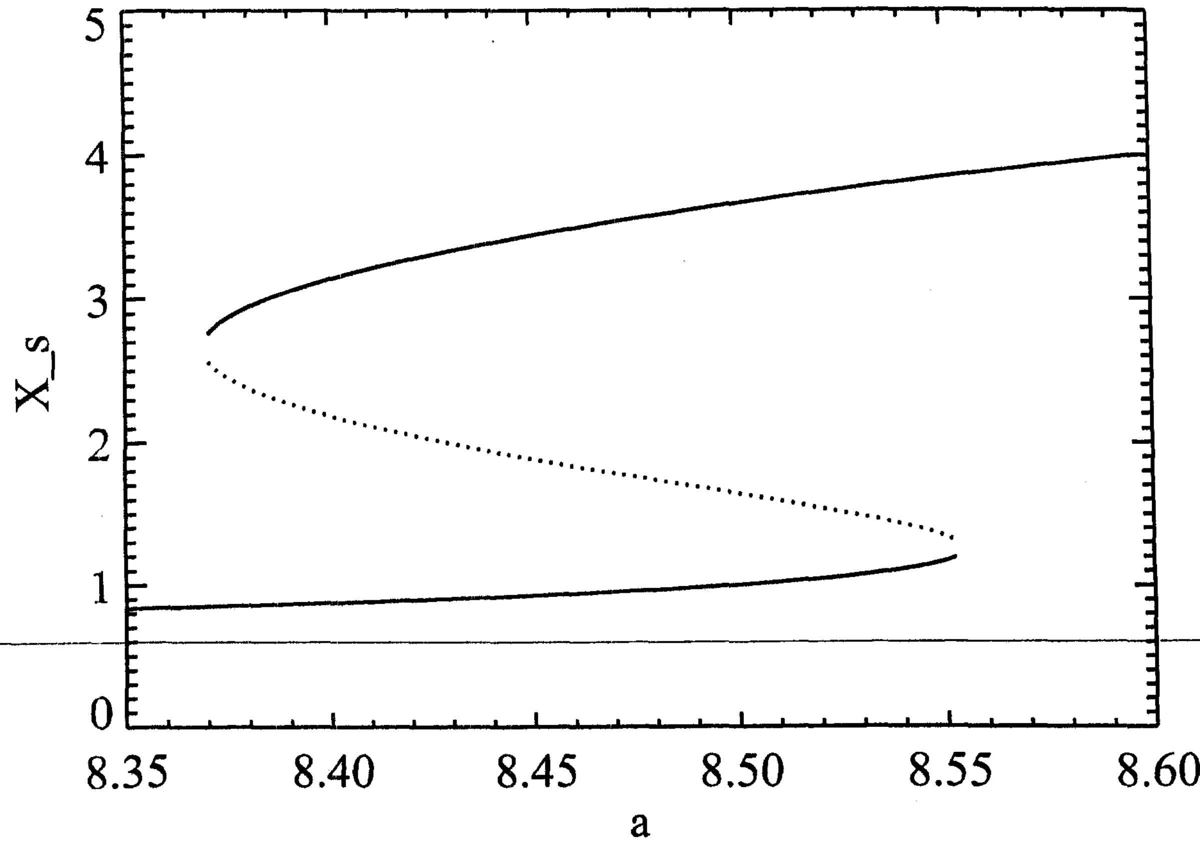

Fig. 5 Konno and Lomdahl $\times \frac{1}{2}$

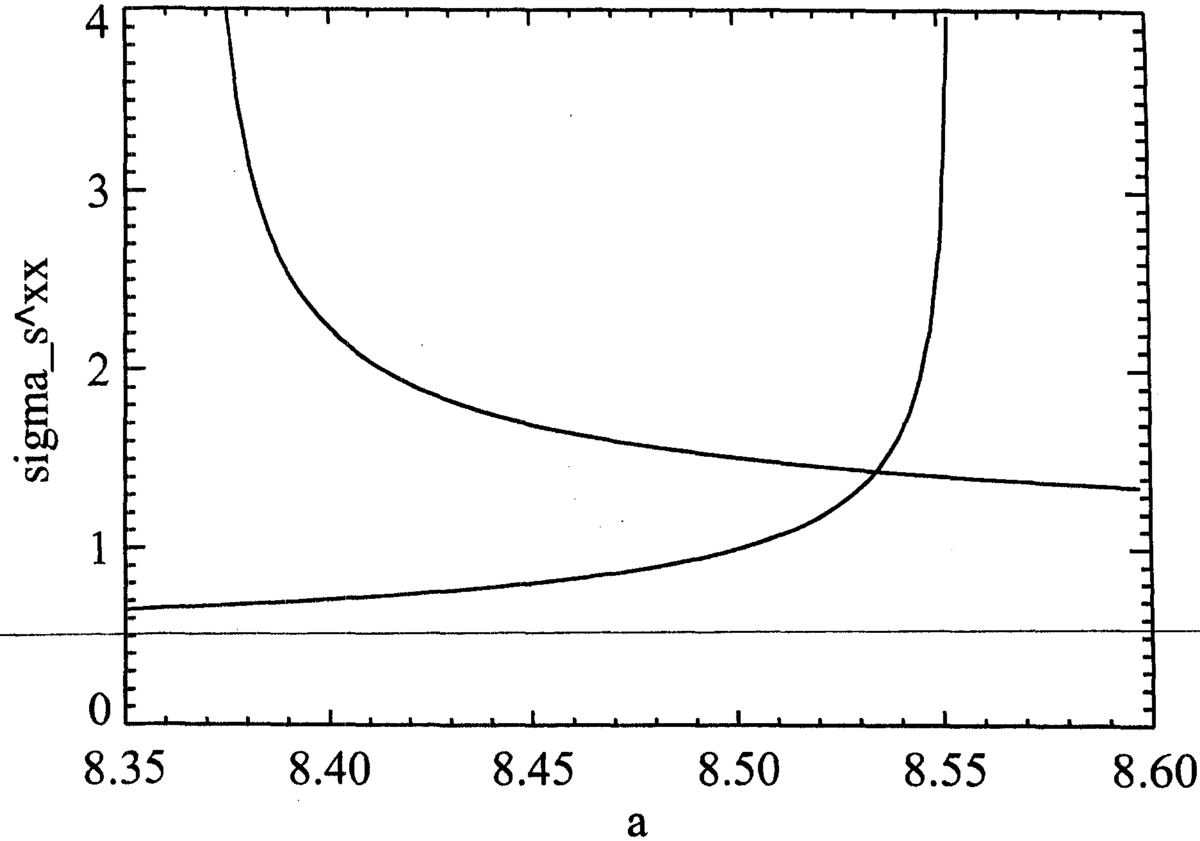

Fig. 6 Konno and Lomdahl $\times \frac{1}{2}$

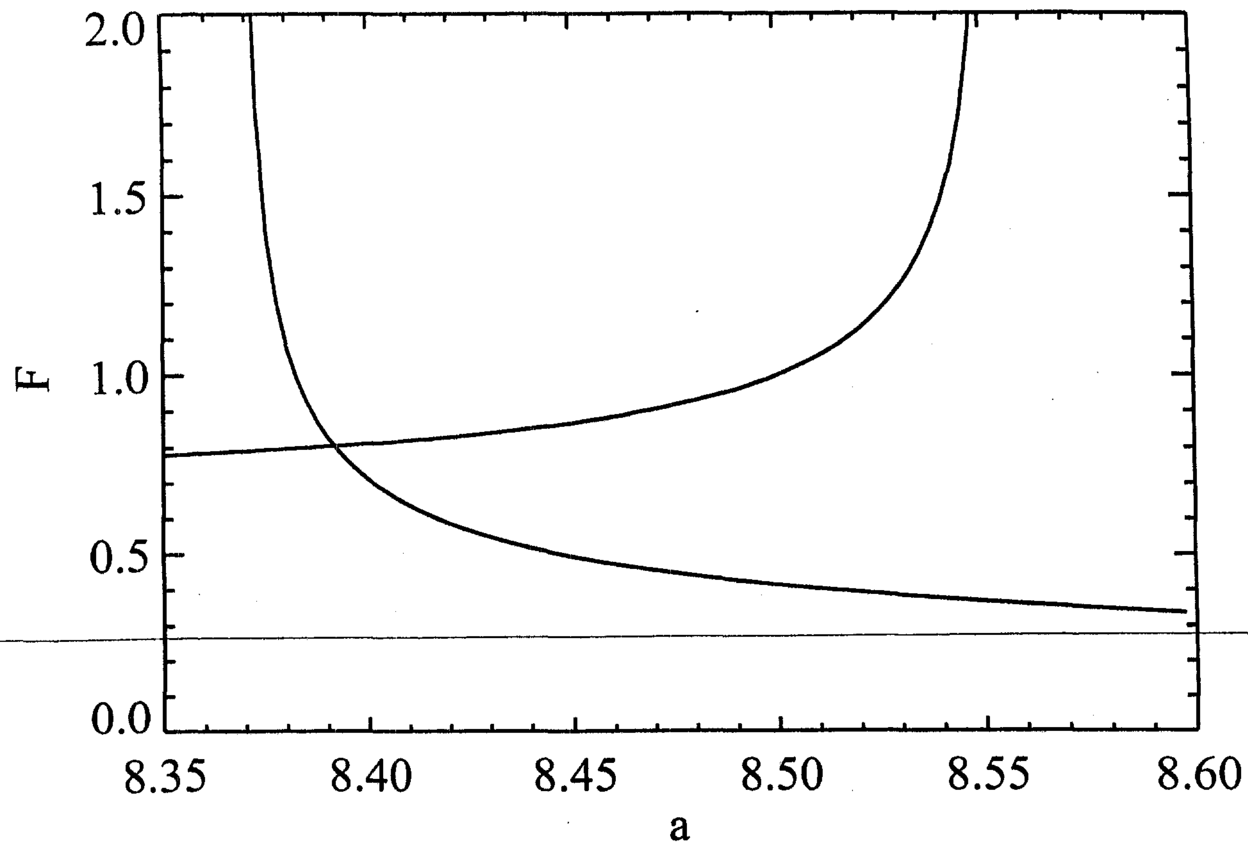

Fig. 7 Konno and Lomdahl $\times \frac{1}{2}$

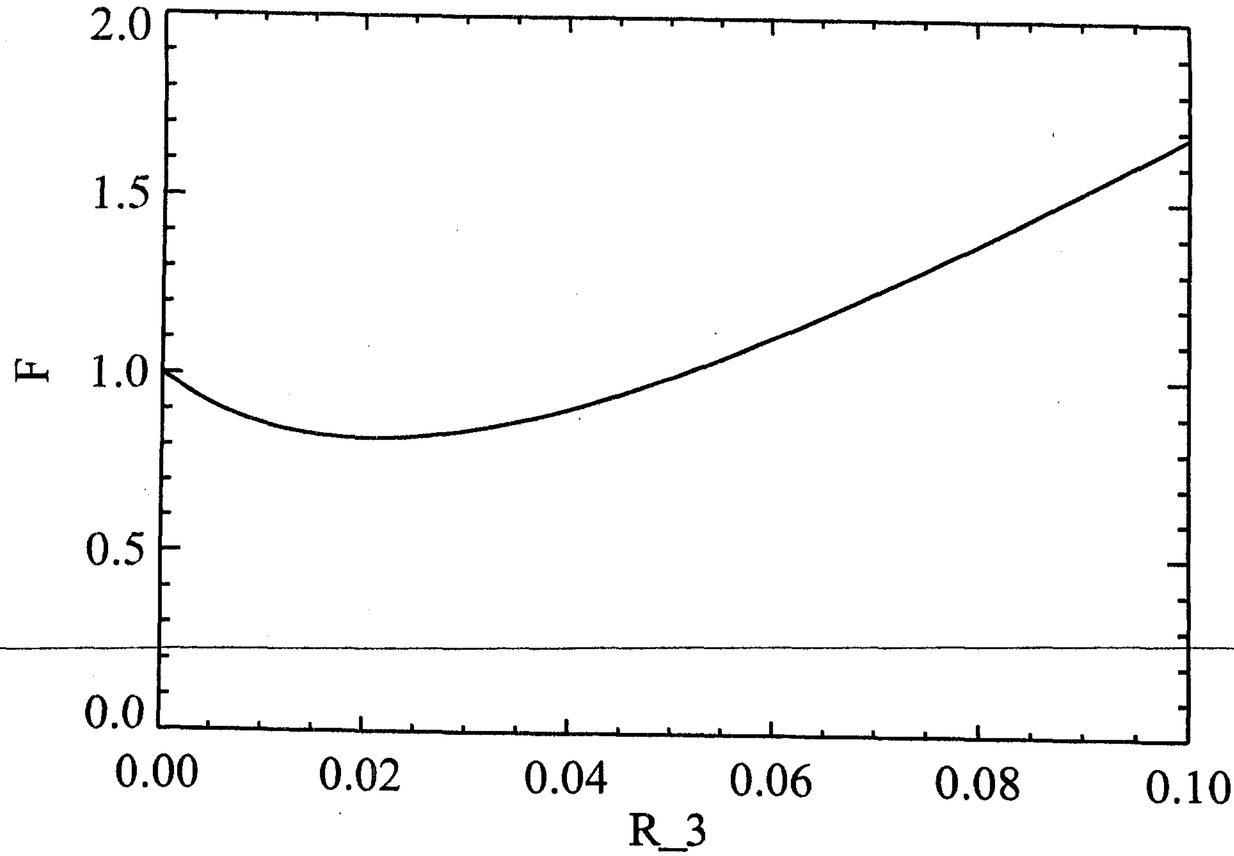

Fig. 8(a) Konno and Lomdahl $\times \frac{1}{2}$

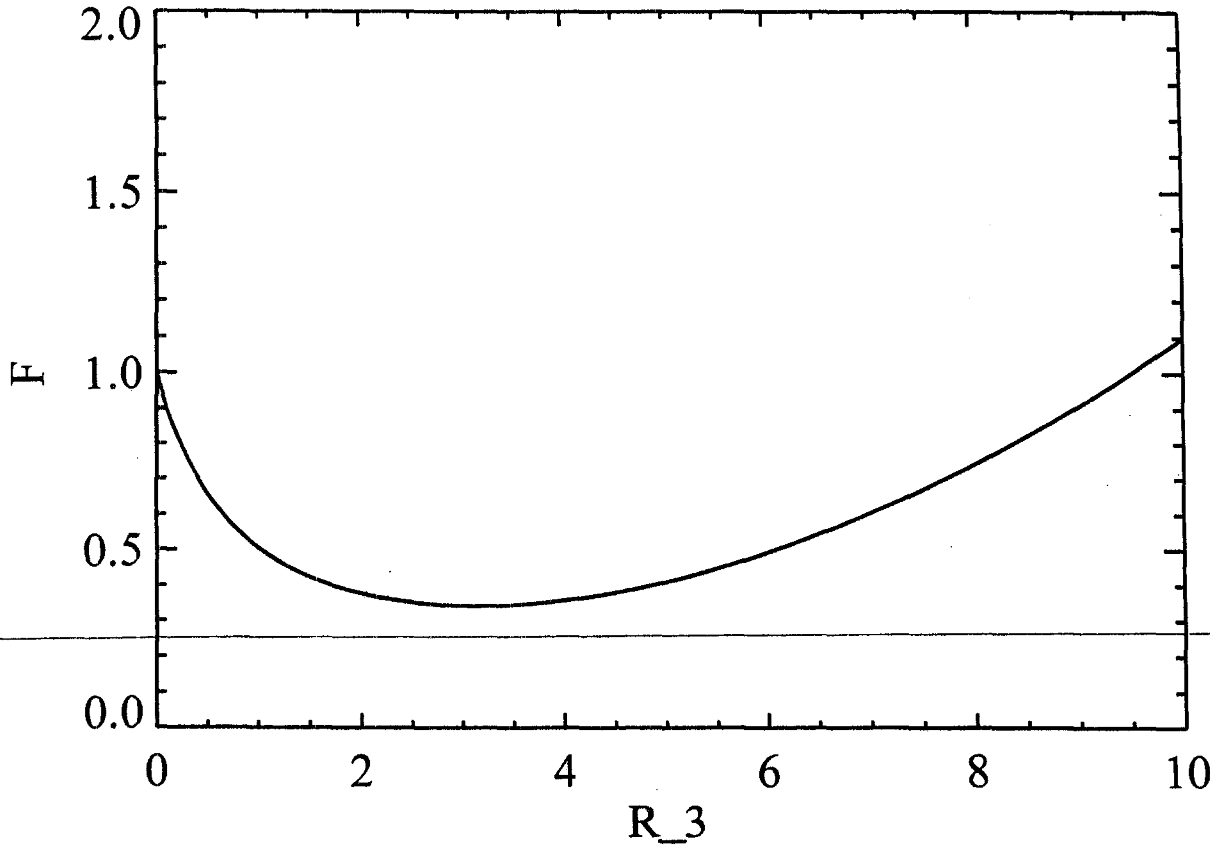

Fig. 8 (b) Konno and Londahl × $\frac{1}{2}$

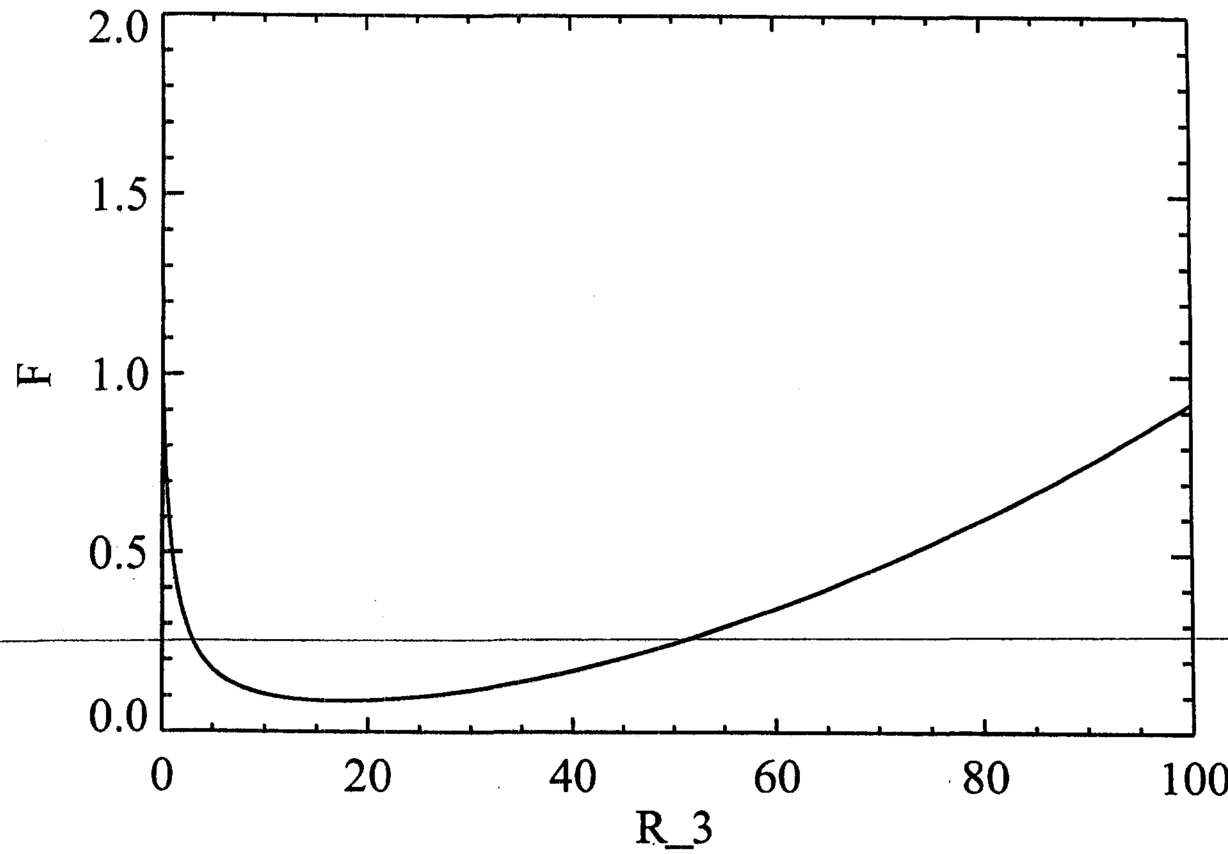

Fig. 8 (c) Konno and Lomdahl $\times \frac{1}{2}$